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A Note on Linear Transformations Which Leave Controllable Multiinput Descriptor Systems Controllable

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ABSTRACT

Consider a generalized linear dynamical system $E\dot{x} = Ax + Bu$, where $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$, and E, A, B are matrices of appropriate sizes with entries in $\mathbb{C}$. This system, or the matrix triple (E, A, B), is called controllable if $\det(\alpha E - \beta A)$ is not a zero polynomial in α, β and $(\alpha E - \beta A, B)$ is of full rank for all $(\alpha, \beta) \in \mathbb{C} \setminus \{(0,0)\}$. Let f be a linear transformation on $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$, the linear space of all matrix pairs (A, B). In an earlier paper, Mehrmann and Krause attempted to prove that, if f is of the form $X \mapsto UXV$, and rank $f(\alpha E - \beta A, B) = n$ for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and all controllable systems (E, A, B), then U, V are nonsingular matrix with V in some lower block triangular form. In this paper, we correct an error contained in this result and discuss whether the corrected result can be generalized in such a way that no restrictions are placed on the form of f.

1. INTRODUCTION

Linear generalized dynamical systems, or descriptor systems, are described in *generalized state space form* by

$$E\dot{x} = Ax + Bu, \tag{1.1}$$

where E, $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$, $C \in \mathbb{F}^{p\times n}$ are constant matrices, and $u \in \mathbb{F}^m$, $x \in \mathbb{F}^n$ are time dependent vectors, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and

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$n, m \in \mathbb{N}$. These systems provide a natural generalization of the class of state space systems in *standard form*:

$$\dot{x} = Ax + Bu. \tag{1.2}$$

The system (1.1), or simply the triple (E, A, B), is called *solvable* or *admissible* if $\alpha E - \beta A$ is a regular pencil, i.e., the genericity condition for the homogeneous polynomial

$$\det(\alpha E - \beta A) \not\equiv 0 \tag{1.3}$$

in (α, β) holds. In other words, $\det(\alpha E - \beta A)$ is not identically zero for $\alpha, \beta \in \mathbb{C}$ (notice that α, β are complex scalars whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

An admissible system (E, A, B) is called *controllable* if

$$\operatorname{rank}(\alpha E - \beta A, B) = n \qquad \forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}. \tag{1.4}$$

A system (A, B) in standard form is also called *controllable* if the generalized system (I, A, B) is controllable, i.e., if

$$\operatorname{rank}(A - \lambda I, B) = n \qquad \forall \lambda \in \mathbb{C}. \tag{1.5}$$

For linear systems (1.1) and (1.2) a problem of particular interest is to find those linear mappings over the real or complex field which in some sense "preserve" controllability. A problem of this kind was mentioned first by Mehrmann and Krause [3] and then by Li, Rodman, and Tsing [2] as a linear preserver problem. In [3], Mehrmann and Krause proved the following result (where the underlying field is $\mathbb{C}$):

THEOREM 1.1 [3, Theorem 3.7]. *Let*

$$f : \mathbb{C}^{n \times (n+m)} \to \mathbb{C}^{n \times (n+m)}$$
$$: X \mapsto UXV,$$

where $U \in \mathbb{C}^{n \times n}$, $V \in \mathbb{C}^{(n+m) \times (n+m)}$, *and* $m \leqslant n$. *Assume that for any controllable system*

$$\dot{x} = Ax + Bu$$

the transformed system

$$\dot{x} = \tilde{A}x + \tilde{B}u$$

is also controllable, where

$$(\tilde{A}, \tilde{B}) = U(A, B)V.$$

Then U is nonsingular and

$$V = \begin{pmatrix} tU^{-1} & 0 \\ F & R \end{pmatrix}$$

with $R \in \mathbb{C}^{m\times m}$ nonsingular and $F \in \mathbb{C}^{m\times n}$ arbitrary (where the scalar t is arbitrary when $n = 1$ and nonzero otherwise).

Then they remarked that the assumption $m \leqslant n$ in the above theorem can be relaxed to arbitrary m. By using this theorem without restrictions on n, m, they worked towards their ultimate goal: to characterize "a special class of linear transformations that leave the controllability of general(ized) linear systems invariant" in the sense given by the following statement.

STATEMENT 1.2 [3, Corollary 3.30]. *Let*

$$f : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$$
$$: X \mapsto UXV,$$

where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{(n+m)\times(n+m)}$, and $m \leqslant n$, such that $\operatorname{rank}[U(\alpha E - \beta A, B)V] = n$ *for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and for any controllable system*

$$E\dot{x} = Ax + Bu.$$

Then U is nonsingular and

$$V = \begin{pmatrix} Q & 0 \\ F & R \end{pmatrix}$$

with $Q \in \mathbb{C}^{n\times n}$ nonsingular, $R \in \mathbb{C}^{m\times m}$ nonsingular, and $F \in \mathbb{C}^{m\times n}$ arbitrary.

Unfortunately, both Statement 1.2 and its proof are wrong. The basic objective of this paper is to correct the error contained in Statement 1.2. We shall also attempt to generalize the result of the corrected version of Statement 1.2 to general linear operators f.

The following notation is used throughout the paper. The symbols n, m always denote positive integers. 0 may denote the zero scalar, a zero vector, a zero matrix, or a null subspace. I_q is the $q \times q$ identity matrix. In expressions of the form (E, A, B) or (A, B), E, A are always square matrices of order n and B a matrix of size $n \times m$. $\{e_1^{(q)}, e_2^{(q)}, \dots, e_q^{(q)}\}$ are the column vectors which form the usual basis of $\mathbb{C}^q$. $E_{ij}^{(p\times q)}$ denotes the $p \times q$ matrix whose (i, j)th entry is 1 and all other entries zero, i.e., $E_{ij}^{(p\times q)} = e_i^{(p)}(e_j^{(q)})^t$. When appropriate, we shall write I without subscript to denote an identity matrix of a suitable order, and drop the superscripts indicating dimensions in the above notation.

2. CORRECTION OF THE STATEMENT

THEOREM 2.1. *Let*

$$f : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$$
$$: X \mapsto UXV,$$

where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{(n+m)\times(n+m)}$. *Then* $\operatorname{rank}[U(\alpha E - \beta A, B)V] = n$ *for all* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *and for any controllable system* $E\dot{x} = Ax + Bu$, *if and only if U is nonsingular and either* (a) *V is nonsingular if* $n \geqslant 2$, *or* (b) *the last m rows of V are linearly independent if* $n = 1$.

Note that we do not impose the condition $n \geqslant m$ here, as was done in Theorem 1.1.

Proof. (a): Suppose $n \geqslant 2$. The sufficiency part is obvious, because the rank of a matrix is invariant under left or right multiplication by a nonsingular matrix. Consider the necessity part. Suppose V is singular and $(a^t, b^t)V = 0$, where $(a^t, b^t) = (a_1, \dots, a_n, b_1, \dots, b_m) \neq 0$. If $a = 0$, let

$$E = I, \qquad A = E_{12} + E_{23} + \cdots + E_{n-1,n}, \quad \text{and} \quad B = e_n b^t.$$

Then $E\dot{x} = Ax + Bu$ is a controllable system, but $\operatorname{rank}[U(A, B)V] \leqslant \operatorname{rank}[(A, B)V] < n$. Hence we must have $a \neq 0$. Let $E \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $a^t E^{-1} = e_1^t$. Let also $A = (E_{12} + E_{23} + \cdots + E_{n-1,n})E$ and $B = e_1 b^t + E_{n1}$. Then $I\dot{x} = AE^{-1}x + Bu$ and in turn $E\dot{x} = Ax + Bu$ are controllable systems, but $\operatorname{rank}[U(E, B)V] \leqslant$

$\text{rank}[(E, B)V] < n$ because the first row of (E, B) is (a^t, b^t). Thus contradiction arises unless the equation $(a^t, b^t)V = 0$ has only the trivial solution. Therefore V is necessarily nonsingular. We use the same idea to show the nonsingularity of U. First we reduce the problem by assuming $V = I_{n+m}$. This is justified by the sufficiency part. Now suppose U is singular. Then there exists a vector $b \neq 0$ such that $Ub = 0$. Let U_1 be a nonsingular matrix such that $U_1 b = e_n$. Let $A = U_1^{-1}(E_{12} + E_{23} + \cdots + E_{n-1,n})$ and $B = U_1^{-1}E_{n1} = (b, 0, \ldots, 0)$. Then $I\dot{x} = U_1 Ax + U_1 Bu$ and in turn $U_1^{-1}\dot{x} = Ax + Bu$ are controllable systems, but $\text{rank}[U(A, B)] = \text{rank}(UA, UB) = \text{rank}(UA, 0) \leqslant \text{rank } A < n$, which is a contradiction. Thus U is nonsingular.

(b): Suppose $n = 1$. In this case an admissible system (E, A, B) is controllable if and only if $B \neq 0$. Hence f satisfies the given hypothesis if and only if $U(0, B)V \neq 0$ whenever $B \neq 0$. The result now follows. ■

Thus Statement 1.2 is false, because it implies that V must be block lower triangular, while Theorem 2.1 tells us that V can be, for example, any nonsingular matrix. In the proof of Corollary 3.30 in [3], it seems that the authors mistakenly assumed that (2.1) ⇒ (2.2) in the following:

$$\text{rank}[U(\alpha I - A, B)V] = n \qquad \forall \alpha \in \mathbb{C}; \tag{2.1}$$

$$U(A, B)V \text{ is controllable (in the standard case)}. \tag{2.2}$$

A counterexample to this logical implication is given as follows. Let $m = n$,

$$U = I, \qquad V = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$A = E_{12} + E_{23} + \cdots + E_{n-1,n}$, and $B = E_{nn}$. It is easy to see that $\text{rank}(\alpha I - A, B) = n$ for all α, and therefore (2.1) is satisfied. However, $U(A, B)V = (B, A)$ is not a controllable system, because

$$\text{rank}(I - B, A) = n - 1 < n.$$

In fact, even if Statement 1.2 is true, what Mehrmann and Krause can infer from Theorem 1.1 is merely that *Statement 1.2 is true under the assumption* $m \leqslant n$, because their reason for relaxing this assumption is invalid. In the remark following [3, Theorem 3.7], Mehrmann and Krause argued in the following way: For each $B \in \mathbb{C}^{n \times m}$ there exists a nonsingular Q such that the rightmost m columns of BQ are zero. Therefore each system $E\dot{x} = Ax + Bu$ can be reduced to a system $E\dot{x} = Ax + BQu$ which is essentially one with $m \leqslant n$. So they claimed that we may relax the assumption by considering $\tilde{f}: X \mapsto f(X(I_n \oplus Q))$. However, this argument does not work, because the matrix Q is not uniform for all B.

In the sequel we will not require that $m \leqslant n$.

3. GENERALIZATIONS

Having corrected Statement 1.2, we may ask (as Mehrmann and Krause did): in general, what linear maps preserve the controllability of (generalized or standard) linear systems? That is, what are the conclusions of Theorem 1.1 and 2.1 if we remove the assumption $f: X \mapsto UXV$ from these theorems? Naturally, one expects the same conclusions, which we shall formulate in the following.

CONJECTURE 3.1. *Suppose $f: \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$ is a linear map. Then $f(A, B)$ is controllable whenever $(A, B) \in \mathbb{C}^{n\times(n+m)}$ is controllable if and only if*

$$f(X) = UXV$$

for any $X \in \mathbb{C}^{n\times(n+m)}$, with U nonsingular and

$$V = \begin{pmatrix} tU^{-1} & 0 \\ F & R \end{pmatrix}, \tag{3.1}$$

where R is nonsingular, F is arbitrary, and the scalar t is arbitrary when $n = 1$ and nonzero otherwise.

CONJECTURE 3.2. *Let $n \geqslant 2$ and $f: \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$ be linear. Then* rank $f(\alpha E - \beta A, B) = n$ *for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and for any controllable system $E\dot{x} = Ax + Bu$, if and only if*

$$f(X) = UXV$$

for all $X \in \mathbb{C}^{n\times(n+m)}$, with U, V nonsingular. (Note that we have no need to consider the case $n = 1$, in which all linear operators on $\mathbb{C}^{n\times(n+m)}$ are of the form $f(X) = UXV$.)

Unfortunately, the answers to these two conjectures are negative. The first conjecture is false because the possibility of adding scalar multiples of

$(I, 0)$ has been overlooked. It is a well-known fact that if (A, B) is controllable, then $(A + \mu I, B)$ is also controllable for any $\mu \in \mathbb{C}$. Hence for any linear functional $\mu : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}$, the linear function defined by $f(X) = UXV + \mu(X)(I, 0)$ for all $X \in \mathbb{C}^{n\times(n+m)}$, where U is nonsingular and V satisfies (3.1), must map controllable matrix pairs to controllable matrix pairs. However, a linear function f defined in this way could be singular, which is in conflict with the conclusion of Conjecture 3.1 when $n > 1$. An example of this kind of f is given by $f(A, B) = (A - (1/n)(\mathrm{tr}\, A)I, B)$, where tr denotes the trace of a matrix. The following is a corrected version of this conjecture.

THEOREM 3.3 [1, Theorem 2.1]. *Suppose $f : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$ is a linear map. Then $f(A, B)$ is controllable whenever $(A, B) \in \mathbb{C}^{n\times(n+m)}$ is controllable if and only if*

$$f(X) = UXV + \mu(X)(I, 0)$$

for any $X \in \mathbb{C}^{n\times(n+m)}$, with $\mu : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}$ a linear functional, U nonsingular, and V a nonsingular matrix of the form

$$\begin{pmatrix} tU^{-1} & 0 \\ F & R \end{pmatrix}$$

where $t \in \mathbb{C} \setminus 0$, R is nonsingular, and F is arbitrary.

For the second conjecture, we can also find a counterexample in which f is not injective.

EXAMPLE 3.4. Suppose $n = 2$, $m = 1$, and $f : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$ takes the following form:

$$f\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & 0 \\ 0 & b_1 & b_2 \end{pmatrix}.$$

For any linear system $E\dot{x} = Ax + Bu$, there exist $\alpha, \beta \in \mathbb{C} \setminus \{(0, 0)\}$ such that $\mathrm{rank}(\alpha E - \beta A) < n$. Hence the system is controllable only if $B \neq 0$. In this case the definition of f shows that $f(\alpha E - \beta A, B)$ is of full rank. Hence f preserves controllability in the sense of Conjecture 3.2.

However, by Theorem 3.3, we can show that Conjecture 3.2 is correct if we impose an additional condition on it.

THEOREM 3.5. *Let* $f : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$ *be such that* rank $f(E_0, 0) = n$ *for some nonsingular matrix* E_0. *Then* rank $f(\alpha E - \beta A, B) = n$ *for all* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$, *and for any controllable system* $E\dot{x} = Ax + Bu$, *if and only if*

$$f(X) = UXV$$

for all $X \in \mathbb{C}^{n\times(n+m)}$, *with* U, V *nonsingular.*

Proof. The sufficiency part is obvious. We need only to prove the necessity part. Let M, N be nonsingular matrices of appropriate sizes such that $M f(E_0, 0)N = (I, 0)$. By considering the linear mapping $\tilde{f}$ defined by $\tilde{f}(A, B) = M f(AE_0, B)N$ for all $(A, B) \in \mathbb{C}^{n\times(n+m)}$, we may assume that $f(I, 0) = (I, 0)$. Now for any controllable (in the standard case) system (A, B), $I\dot{x} = Ax + Bu$ is also controllable in the generalized case. Therefore, by the given hypothesis, $n = \operatorname{rank} f(\alpha I - A, B) = \operatorname{rank}[f(A, B) - \alpha(I, 0)]$ for all $\alpha \in \mathbb{C}$. This means, according to Theorem 3.3,

$$f(X) = UXV + \mu(X)(I, 0)$$

for any $X \in \mathbb{C}^{n\times(n+m)}$, where U, V, μ satisfy the conditions given in Theorem 3.3. By the sufficiency part of Theorem 3.3, we may assume that $U = I_n$ and $V = I_{n+m}$, and

$$f(A, B) = (A + \mu(A, B)I, B) \tag{3.5.1}$$

for any $(A, B) \in \mathbb{C}^{n\times(n+m)}$. In this case the property $f(I, 0) = (I, 0)$ may no longer prevail, but $f(I, 0)$ is still a nonzero scalar multiple of $(I, 0)$ [so that $1 + \mu(I, 0) \neq 0$]. When $n = 1$, this means

$$f(A, B) \equiv (A, B)\begin{pmatrix} 1 + \mu(1,0) & 0 \\ \sum_{i=1}^{m} \mu(0, e_i^t)e_i & I \end{pmatrix},$$

and we are done. Now suppose $n \geqslant 2$. We are going to show that $\mu = 0$, and the result will follow. By (3.5.1) and the given hypothesis we have

$$\operatorname{rank}(A + \mu(A, B)I, B) = n$$
$$\text{for any controllable system } E\dot{x} = Ax + Bu. \tag{3.5.2}$$

Putting $E = E_{12} + E_{23} + \cdots + E_{n-1,n}$ (the square matrix whose superdiagonal entries are 1's and elsewhere 0), $A = I$, and $B = kE_{n1}$ ($k \neq 0$), we get

$$n = \operatorname{rank}\big(\big[1 + \mu(I,0) + k\mu(0, E_{n1})\big]I, kE_{n1}\big)$$

$$= \operatorname{rank}\big(\big[1 + \mu(I,0) + k\mu(0, E_{n1})\big]I, E_{n1}\big)$$

for any $k \neq 0$. This is possible only if $\mu(0, E_{n1}) = 0$. By a similar argument we get $\mu(0, B) = 0$ for all B. Hence we may assume μ to be a linear functional $\mu = \mu(A)$ defined on $\mathbb{C}^{n\times n}$. Also, by putting in (3.5.2) $E = E_{12} + E_{23} + \cdots + E_{n,n-1} + \varepsilon E_{nn}$, $A = I - \varepsilon E_{nn}$, and $B = E_{n1}$ ($\varepsilon \in \mathbb{C}$), we get

$$n = \operatorname{rank}\big(\big[1 + \mu(I) - \varepsilon\mu(E_{nn})\big]I - \varepsilon E_{nn}, E_{n1}\big)$$

for any $\varepsilon \in \mathbb{C}$. Thus $\mu(E_{nn}) = 0$, and similarly $\mu(E_{ii}) = 0$ for each i. Now consider $E = E_{12} + E_{23} + \cdots + E_{n-1,n}$, $A = I + kE_{rs}$, and $B = E_{n1}$, where $k \in \mathbb{C}$ and E_{rs} does not lie on the main diagonal or the superdiagonal. By (3.5.2) we get $n = \operatorname{rank}(A + \mu(A)I, B) = \operatorname{rank}([1 + \mu(I) + k\mu(E_{rs})]I + kE_{rs}, E_{n1})$ for all $k \in \mathbb{C}$. This is possible only if $\mu(E_{rs}) = 0$. Similarly, by considering E^t, A^t, and E_{11} instead of E, A, B, we can also show that $\mu(E_{rs}) = 0$ whenever E_{rs} does not lie on the main diagonal or the subdiagonal. Hence $\mu(E_{ij}) = 0$ for all i, j, and thus $\mu = 0$. ∎

Based on Theorem 3.5, and some results concerning so-called *rank preservers* and *rank-k subspaces* which we shall not discuss here (interested readers may read the survey written by Lim [4, Chapter 2]) the following seems to be believable:

CONJECTURE 3.6. *Unless $n = 2$ and $m = 1$, Conjecture 3.2 is true.*

Another way to define the preservation of controllability of generalized linear systems and to formulate the corresponding linear preserver problem is given as follows.

PROBLEM 3.7. Determine the structure of all linear mappings f defined on $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ such that $f(E, A, B)$ is admissible and respectively controllable whenever (E, A, B) is admissible and respectively controllable.

The rationale for this setting is the following: In the original problem, $f(\alpha E - \beta A, B)$ does not necessarily represent any dynamical system or any matrix pencil generated by a dynamical system, because it is not of the form $(\alpha E' - \beta A', B')$. In other words, although one can blend the components of a system (E, A, B) to form a matrix pencil, it is not always possible to recover any system from the pencil $f(\alpha E - \beta A, B)$. Thus, unlike Theorem 1.1 or Problem 3.7, we have no "transformed system" to speak of, and the scope of analysis is limited to the property "(E, A, B) controllable $\Rightarrow$ rank $f(\alpha E - \beta A, B) \equiv n$" only. In the new setting, however, each $f(E, A, B)$ is a matrix triple and hence represents some linear system.

While the author is not sufficiently familiar with system theory to comment on which setting is physically more sensible, surely both settings are mathematically legitimate. For the new one the author conjectures that all controllability preservers are composed of some common mappings for generalized linear systems, namely the scaling actions and the actions of restricted system equivalence.

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